Exercise Set #4

"Discrete Mathematics" (2025)

Exercise 7 is to be submitted on Moodle before 23:59 on March 17th, 2025

E1. In how many ways can you write $3, 7$, and 12 as a sum of three numbers chosen from the sets $\{1,2,3,5,8\}$, $\{2,3,5,7\}$, and $\{0,2,4\}$ respectively?

E2. Find the sequence generated by the generating function:

(a) $\frac{x^3}{(1+x)^2}$.

(b) $\frac{1+x+x^2}{(1-x)^2}$.

E3. Determine the generating function of the following sequences and write them as a closed expression.

(a) $(a_0, a_1, a_2, a_3, a_4, \ldots) = (1, 3, 5, 7, 9, \ldots)$;

(b) $(a_0, a_1, a_2, a_3, a_4, \ldots) = (1, 0, 1, 0, 1, \ldots)$;

(c) $a_n = n^2 + 1$ for each $n \in \mathbb{N}$;

E4. Prove the following using generating polynomials.

$$\sum_{k=q}^{n} \binom{n}{k}\binom{k}{q} = 2^{n-q}\binom{n}{q}.$$

E5. Let $\{a_n\}_{n\geq 0}$ denote the sequence of non-negative integers given by $a_0 = 1$ and $a_{n+1} = 2a_n + n$.

(a) Let

$$A(x) = \sum_{n\geq 0} a_n x^n$$

be the generating function associated with $\{a_n\}_{n\geq 0}$. Find $A(x)$.

(b) Give an explicit expression for a_n.

E6. Suppose $n, k_1, k_2, \ldots, k_r \in \mathbb{Z}^{\geq 0}$ such that $\sum_{i=1}^{r} k_r = n$. We use the following notation for multinomials:

$$\binom{n}{k_1, k_2, k_3, \ldots, k_r} = \frac{n!}{k_1!k_2!\ldots k_r!}.$$

(a) Prove that

$$\binom{n}{k_1, k_2, \ldots, k_r} = \binom{n-1}{k_1 - 1, k_2, \ldots, k_r} + \binom{n-1}{k_1, k_2 - 1, k_3, \ldots, k_r} + \cdots \binom{n-1}{k_1, k_2, \ldots, k_r - 1}.$$

Compare this with Proposition 1.11 from the lecture about binomial coefficients.

(b) Show that

$$\sum_{k_1+k_2+\ldots+k_m=n} \binom{n}{k_1, k_2, \ldots k_m} = m^n$$

E7. (**Exercise to submit**)

(a) We toss a coin n times. Compute the probabilities that the number of tails equals $0, 1,$ or 2 modulo 3. Where do these probabilities converge as n goes to infinity?

(b) Using generating functions, show that

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k}^2 = (-1)^n \binom{2n}{n}.$$

Hint: Consider $G(x) = \left(1 - x^2\right)^{2n}$.